Strategy-Proofness in the Spatial Model of Politics

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Outline of the IPDM Talk

· Generalities

· Equivalence of Strategy-Proofness and Coalitional Strategy-Proofness

· Decomposability

· The Spatial Model of Politics

· Strategy-proofness, Pareto Optimality

· Self Dual Cones and Jordan Algebras

Notations and Generalities

· A society (group) described by a finite set $N = \{1, 2, ..., n\}$ of individuals

· A set X of feasible alternatives.

· Each individual $i \in N$ is described by his/her preference R_i over X, which is assumed to be a complete preorder We will denote respectively by P_i and I_i the strict preference and the indifference relations induced by R_i. Sometimes, we will represent a preference R_i by a utility function U_i

· A profile of preferences is a vector $\pi \equiv (R_1, R_2, ..., R_n)$ describing the preferences of each individual in the society. If π is a profile of preferences and $S \subseteq N$ is a nonempty subset of individuals, then π_S denotes the subprofile $(R_i)_{i \in S}$; when $S = N \backslash \{i\}$ for some $i \in N$, we denote π_{-i} for π_S. If π and π' are two profiles of preferences and $S \subseteq N$, then $\pi'' \equiv \left(\pi_S, \pi'_{N \backslash S}\right)$ denotes the profile such that $\pi''(i) = \pi(i)$ if $i \in S$ and $\pi''(i) = \pi'(i)$ if $i \notin S$.

· *Let* Π *be a subset of profiles. A social choice mechanism with domain* Π *is a mapping* C *from* Π *into* X*.*

· If Π consists of all possible profiles, the domain is said to be *universal.* Otherwise, it is said to be *restricted.* The notion of domain is central in our paper as the results are driven

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by assumptions which will be formulated on the domain. We limit to Cartesian domains i.e. domains such that $\Pi = \prod_{i=1}^{n} D_i$ where for all $i \in N$, D_i is a nonempty subset of complete preorders over X.

The social choice mechanism reflects the aspirations and properties that this society wants to take into account to proceed in selecting a social alternative. The input of such mechanism is a profile of preferences. This means that once we know the diversity of opinions in the society, conflicts but also areas of agreement, we have, in principle, everything needed to pick up a compromise. To operate, the mechanism needs this input, but in most cases, this input is not known or verifiable with certainty by all members of the society.

· *A social choice mechanism C with domain Π is manipulable by individual i at profile π if there exists $R_i' \in D_i$ such that $C(\pi_{-i}, R_i') P_i C(\pi)$. A social choice mechanism C with domain Π is strategyproof if there is no individual i and no profile $\pi \in \Pi$ such that C is manipulable by i at π.*

This property reflects the necessity to provide incentives to individuals to make sure that they report the right information. Strategyproofness is a strong form of incentive compatibility as it requires the existence of dominant strategies. From the perspective of constructing the social choice mechanism, this property acts as a constraint in the design of the rule.

· Some few more definitions and notations are needed. *From now on, we limit our attention* to the case where $D_i \equiv D$ for all $i \in N$. Let $D^* \equiv \cup_{x \in X} D_x$ where :

$$D_x \equiv \{R \in D : xPy \text{ for all } y \in X \setminus \{x\}\}$$

D_x is the set of preferences for which the alternative x is uniquely best. Finally, let $X^* \equiv \{x \in X \text{ such that } D_x \neq \varnothing\}$: an alternative x is in X^* if there exists an admissible preference with x on top. Let C^* be the restriction of C to the subdomain $\Pi^* \equiv (D^*)^n$. Let

$$C(\Pi) \equiv \{x \in X : x = C(\pi) \text{ for some } \pi \in \Pi\}$$

be the range of the mechanism C.

· **Lemma 1** *Let C be a strategyproof social choice mechanism with domain D. For all $\pi \in D^n$ and all $x \in C(\Pi)$, if $R_i \in D_x$ for all $i \in N$, then $C(\pi) = x$.*

· *A social choice mechanism C with domain Π is regular if $C(\Pi) \subseteq X^*$.*

To the best of our knowledge, this property is new. It requires that the range of the mechanism is contained in the subset of alternatives which appear on the top of an admissible preference. It is certainly controversial in any environment where an alternative which could be considered as a good social compromise is disregarded simply because at best, it appears on second position in any individual preference. In most of this paper, we will consider environments where the property of regularity does not raise any problem. The following simple lemma will be useful.

· *Let C be a strategyproof and regular social choice mechanism with domain D. Then,* $C(\Pi) = C^*(\Pi^*)$.

· *A social choice mechanism C with domain D^n satisfies the modified strong positive association property if for all $\pi \in D^n$, all $i \in N$* and all $x \in C(\Pi)$, *if* $C(\pi) = x$ and $R_i' \in D$ is such that $xP_i'y$ *for all* $y \in C(\Pi)\setminus\{x\}$ such that xR_iy, then $x = C(\pi_{-i}, R_i')$.

· **Lemma 3** *A strategyproof social choice mechanism C with domain Π satisfies the modified strong positive association property.*

The notion of strategyproofness describes individual incentives to report the truth. The next notion deals with the behavior of coalitions.

· *A social choice mechanism C with domain Π is manipulable by coalition S at profile π if there exists $\pi' \in \Pi$ such that $C(\pi'_S, \pi_{N\setminus S}) P_i C(\pi)$* for all $i \in S$. *A social choice mechanism C with domain Π is coalitional strategyproof if there is no coalition S and no profile $\pi \in \Pi$ such that C is manipulable by S at π.*

Coalitional strategy proofness is obviously more demanding than strategyproofness.

· Given a social choice mechanism C with domain D^n, a profile $\pi \in D^n$ and a coalition $S \subseteq N$, we denote by $C_S[\pi_{N\setminus S}]$ the social choice mechanism defined over the subsociety S with domain D^S by :

$$C_S[\pi_{N\setminus S}](\pi'_S) = C(\pi'_S, \pi_{N\setminus S}) \text{ for all } \pi'_S \in D^S$$

The range of the mechanism $C_S[\pi_{N\setminus S}]$ will be denoted $A_S[\pi_{N\setminus S}]$: it describes the set of alternatives (options) attainable by coalition S given the subprofile $[\pi_{N\setminus S}]$ of reports by individuals outside coalition S. These sets , called *option sets* by Laffond (1980) and Barbera and Peleg (1990) will play a critical role in the rest of the paper. For all $i \in N$ and $\pi \in D^n$, the option set of coalition $\{i\}$ will be denoted $A_i[\pi_{-i}]$.

· **Lemma 4** *Let X be a metric space and D be a subset of the set of continuous preferences over X . If C is a strategyproof social choice mechanism with domain D^n then for all $\pi \in D^n$ and all $S \subseteq N$, $A_S[\pi_{N\setminus S}] \cap X^*$ is a closed subset of X^*.*

Proof : Let $\pi \in D^n$, $S \subseteq N$ and $x \in X^*\setminus A_S[\pi_{N\setminus S}]$. We claim that there exists $\varepsilon > 0$ such that :

$$B(x, \varepsilon) \cap A_S[\pi_{N\setminus S}] = \varnothing$$

where denotes the open ball centered on x with radius ε i.e. $B(x,\varepsilon)=\{y\in X : d(x,y)<\varepsilon\}$. Suppose on the contrary that for all $\varepsilon>0$, there exists $z_\varepsilon\in B(x,\varepsilon)\cap X^*$ such that $z_\varepsilon\in A_S\left[\pi_{N\setminus S}\right]$. Since $x\in X^*$, there exists $R^*\in D$ such that $R^*\in D_x$. Let $\pi'\equiv(R^*,R^*,...,R^*)$ $y\equiv C\left(\pi'_S,\pi_{N\setminus S}\right)$. Since $y\neq x$ and $R^*\in D_x$ we deduce : xP^*y . Since preferences are continuous, we deduce that there exists $\delta>0$ such that for all z in the ball $B(x,\delta)$: zP^*y. Select such a $\varepsilon>0$ smaller than δ and $R^\varepsilon\in D$ such that such $R^\varepsilon\in D_{z_\varepsilon}$. Let $\pi^\varepsilon\equiv(R^\varepsilon,R^\varepsilon,...,R^\varepsilon)$ and

$$w=C(\pi^\varepsilon_S,\pi_{N\setminus S})$$

Since $z_\varepsilon\in A_S\left[\pi_{N\setminus S}\right]$, we deduce from Lemma 1 that $w=z_\varepsilon$.

Without loss of generality, let $S\equiv\{1,.....,s\}$ where $s\equiv\#S$ and consider the finite sequence of profiles $(\widetilde{\pi}^j)_{0\leq j\leq s}$ defined as follows :

$$\widetilde{R}^j_i\equiv\begin{cases}R_i \text{ for all } i\notin S\\ R^\varepsilon \text{ for all } i\in\{1,......,j\}\\ R^* \text{ for all } i\in\{j+1,......,s\}\end{cases}$$

Since C is strategyproof, we deduce :

$$C(\widetilde{\pi}^j_S,\pi_{N\setminus S})R^*C(\widetilde{\pi}^{j+1}_S,\pi_{N\setminus S}) \text{ for all } j=0,...,s-1$$

Since $C(\widetilde{\pi}^0_S,\pi_{N\setminus S})=y$ and $C(\widetilde{\pi}^s_S,\pi_{N\setminus S})=w=z_\varepsilon$, we deduce from above and transitivity of R^* that yR^*z_ε, a contradiction to $z_\varepsilon P^*y$ □

Equivalence of Strategy-Proofness and Coalitional Strategy-Proofn

(Le Breton and Zaporozhets (2008))

THE GENERAL RESULT

· We introduce a condition on the domain D of preferences which is sufficient for this equivalence to hold true. This class of domains, that we call *rich domains* hereafter, is defined as follows.

· *A domain* $\Pi=D^n$ *is rich if for all* $R\in D$ *and* $x,y\in X$ *such that* yPx *and* $y\in X^*$, *there exists* $R'\in D$ *such that* $R'\in D_y$ *and for all* $z\neq x$ *such that* xRz, *we have* $xP'z$.

The notion of richness which is stated in Definition 6 is designed to take advantage of the property of modified positive association. Without such domain condition, the property is vacuous as it cannot be used if it is never the case that two profiles are related as described by the premises of the condition.

To be rich, a domain must contain enough preferences. Of course, the universal domain is rich but there are also many restricted domains which meet this richness requirement. Intuitively, when a domain is rich we are able to consider transformations of individual preferences where the positions of two given alternatives are improved in the process. Precisely the alternative y which was best among the two is now best among all and the other one x still strictly dominates the alternatives that it was strictly dominating before but now x also strictly dominates the alternatives belonging to its former indifference curve. This is illustrated on figures 1 and 2 in the case where alternatives are vectors in the two dimensional Euclidean space. On figure 1, we have drawn the upper contour sets of x and y for the preference R. On figure 2, we have reproduced the upper contour set of x for R and drawn, as a dotted curve, the upper contour set of x for R'. The upper contour set of y for R' consists exclusively of y.

It should be transparent from this illustration that for a domain to be rich, we must have enough degrees of freedom to deform preferences. If not, the richness condition is likely to be violated.

· Consider for instance the traditional Euclidean environment popular in formal political science i.e. the setting where X is some Euclidean space $\Re^m$ and D is the subset of Euclidean preferences : a preference R over X is Euclidean if there exists $p \in \Re^m$ such that xRy iff $\| x - p \| \leq \| y - p \|$. The upper contours sets are the spheres centered on p. The set of Euclidean preferences is not rich. Figure 3 in the case where $m = 2$.

· It is important to call the attention on the fact that there are general properties of preferences which preclude the richness condition. For instance, if D is a subset of the set of separable preferences over a Cartesian set of alternatives, then D cannot be rich. To see why, consider the specific case where $X = \Re^2$ and D is the subset of separable preferences with single peaked marginal preferences as defined by Border and Jordan (1983). The domain D contains preferences R such that p is best, yPx and xPz where the respective positions of p, x and y and z are represented on figure 4. The key features of this pattern are that y does not belong to the rectangle generated by x and p and that z belongs to the rectangle generated by x and y. From the definition of D, it follows that any preference $R' \in D$ such that y is on top for R' implies that any alternatives w in the rectangle generated by x and y is preferred to x according to R'. In particular, we have $zR'x$.

· Another illustration of that necessity is provided by the set Q of quadratic preferences considered by Border and Jordan (1983) defined in the case where X is some Euclidean space

$\Re^m$. A preference R is quadratic if there exists $p \in \Re^m$ and a symmetric and positive definite matrix $A = (a_{ij})_{1 \leq i,j \leq m}$ such that xRy iff $(x-p)^t A(x-p) \leq (y-p)^t A(y-p)$. The set of Euclidean preferences considered earlier is a subset corresponding to the case where A is the identity matrix. Border and Jordan prove that when $D = Q$, if C is strategyproof and onto, then C is dictatorial. The set D is not rich. Upper contour sets are now ellipses whose axis can be chosen arbitrarily but this gain in flexibility is not enough to obtain richness. For instance, if we consider a situation like the one depicted on figure 3, then any ellipse centered on y and passing through x will have the property that the symmetric image of x through y on the ellipse will be outside the original circle if y is distant enough from x.

· In evaluating the scope of validity of the richness condition, it is tempting to infer from the above examples that any "very small" domain of preferences is likely to violate the condition of richness; in these examples, preferences are described by a finite number of parameters and this is too much restrictive to reach the desired flexibility. We can find domains disproving this suspicion.

· Finally, it is important to call the attention on the fact that in the premises of the definition of a rich domain, we only require the existence of this new "lifted" profile when $y \in X^*$. If $X \neq X^*$, this can make an important difference. To illustrate this point, consider the case where the set X of alternatives is the unitary m-dimensional unitary simplex :

$$\left\{ x \in \Re^m_+ : \sum_{k=1}^{m} x_k = 1 \right\}$$

and D *is the set of linear preferences*: a preference R is linear if there exists a vector $a \in \Re^m$ such that xRy iff $\langle a, x \rangle \geq \langle a, y \rangle$. In such a case, it is easy to show that X^* is the set of vertices of the simplex. However, the domain is not rich as the definition of richness demands that if xRz for some z, then $xP'z$ for the new preference R'. Given the linearity of indifference contour sets, it is not possible to alter weak preferences into strict preferences. This domain is important as it represents social choice mechanisms where chance is used in addition to individual preferences to select a social alternative: the set of alternatives X is the set of lotteries on a set of m pure alternatives. The property of linearity for individual preferences is equivalent to the requirement that preferences satisfy the von Neumann-Morgenstern axioms. In this setting, pioneered by Gibbard (1977) and further explored by Barbera (1979a) and others, a lot is known upon the class of strategyproof social choice mechanisms. Not all them are of course coalitional strategyproof. For instance, when $m = 3$ and $n = 2$, random dictatorship is not coalitional strategyproof : if vertices 1 and 2 are the peaks of the two individuals and vertex 3 is their common second choice, then reporting the truth is not an optimal strategy if their intensity for the second choice is large enough. Barbera (1979b) provides a description of the class of those which were coalitional strategy proof in that class.

· **Theorem** *Let* C be a *regular social choice mechanism with domain* $\Pi = D^n$. *If* D *is rich, then* C *is strategyproof if and only if* C *is coalitional strategyproof.*

Proof : Let C be a strategyproof social choice mechanism on a domain D^n assumed to be rich. We now prove that C is coalitional strategyproof. assume on the contrary that C is not coalitional strategyproof. Then, there exists $S \subseteq N$ and $\pi, \pi' \in D^n$ such that for all $i \in S$:

$$y \equiv C(\pi'_S, \pi_{N \setminus S}) P_i C(\pi) \equiv x$$

Since D is rich and C is regular, there exists $\pi'' \in D^n$ such that for all $i \in S$:

$$R''_i \in D_y \text{ and for all } z \neq x : xR_i z \Rightarrow xP''_i z$$

and for all $i \notin S$:

$$R''_i = R_i$$

Given the construction of π'' and since $C(\pi) = x$, a repeated application of Lemma 3 leads to :

$$C(\pi'') = x \tag{1}$$

On the other hand, note that since C is strategyproof, the restricted social choice mechanism $C_S \left[\pi_{N \setminus S}\right] = C_S \left[\pi''_{N \setminus S}\right]$ is also strategyproof. Since $x \in A_S \left[\pi''_{N \setminus S}\right]$ and $R''_i \in D_y$ for all $i \in S$, we deduce from Lemma 1 that $C_S \left[\pi''_{N \setminus S}\right] (\pi''_S) = C(\pi'') = y$ in contradiction to (1) □

· *A social choice mechanism* C *with domain* D^n *is dictatorial if there exists an individual* $i \in N$ such that for all $\pi \in D^n$ and all $x, y \in C(\Pi)$, if $xP_i y$, then $C(\pi) \neq y$.

A dictatorial social choice mechanism ignores the preferences of all but one individual : the most preferred alternative of this individual, called the dictator, is selected to be the social outcome.

APPLICATIONS

· To illustrate the usefulness of Theorem 1 though a detailed examination of a specific but important class of environments. When a domain of preferences Π is rich, the analysis of the implications of strategyproofness in the construction of social choice mechanisms is considerably simplified as we know that the mechanism is in fact coalitional strategyproof. Note in particular that if a mechanism C is coalitional strategyproof, then it is Pareto efficient over the range $C(\Pi)$ i.e. there does not exist $\pi \in \Pi$ and $x \in C(\Pi) : xP_iC(\pi)$ for all $i \in N$.

· Since Pareto Efficiency put some constraints on the subset of social outcomes that may be considered, this information can be exploited to simplify the analysis of the mechanism C.

Allocation of a Budget Across Several Different Pure Public Goods

· The allocation environment considered in this section has been examined first by Zhou (1991) and is defined as follows. An exogenous monetary budget of size normalized to 1 is to be allocated across m different pure public goods. The set X of alternatives is therefore the unitary m-dimensional unitary simplex :

$$\left\{ x \in \Re^m_+ : \sum_{k=1}^{m} x_k = 1 \right\}$$

· We assume that each individual $i \in N$ has a preference over the m-dimensional positive orthant $\Re^m_+$ which is assumed to be strictly monotonic and strictly convex. The set D is the set of restrictions of such preferences to the set X. It is straightforward to show that a preference R is in D iff its upper contour sets are strictly convex. Theorem 3 stated below holds true for all n and all $m \geq 3$ but for the sake of simplicity, we will limit our investigation to the case where $m = 3$ and $n = 2$. The case where $m = 2$ is considered in the next subsection.

· **Lemma 6** *Let $m = 3$ and $n = 2$. Then the set D of preferences with strictly convex upper contour sets is rich.*

Proof: Let R be a preference in D such that yPx for some $x, y \in X$. Let A be the upper contour set of x with respect to R i.e.

$$A = \{z \in X : zRx\}$$

A is a closed and strictly convex subset of X with $y \in$ Interior A. Let A' be a closed and convex subset of A such that $y \in$ Interior A' and Boundary $A \cap$ Boundary $A' = \{x\}$. The construction of such subset is illustrated on figure 5. The set A' as depicted is not strictly convex. To get a strict convex set A'' contained in A' and containing y, consider the line orthogonal to $[y', x]$ and passing through $\frac{x+y'}{2}$. If we consider two circles centered respectively in t and u, located on this line, on both sides of the segment $[y', x]$ and with radius $\|x-t\| = \|x-u\|$, then the intersection(s) of the circle(s) with either the half plane above or the half plane below the segment $[y', x]$ are contained in A' if $\|x-t\|$ is large enough. Since by construction this subset is strictly convex, the argument is complete. The construction of A'' is illustrated on figure 6.

Let J be the gauge of $(A'' - \{y\})$ with respect to y i.e. the function defined by :

$$J(w) = \underset{w-y\in\lambda(A''-\{y\})}{Inf\ \lambda}$$

It is well known that J is a continuous and convex (here strictly convex) function such that :

$$J(w) = 1 \text{ iff } w \in \text{Boundary } A''$$

Let R' be the preference generated by $-J$. By construction, $R' \in D_y$ and for all $z \neq x$, xRz implies $xP'z$ □

· **Lemma 7** *Let $m = 3$ and $n = 2$. Let C be a strategyproof social choice mechanism over D such that $C(D^n) = X$. Then, for all $x \in X$ and all R_1, $R'_1 \in D_x$, $A_2(R_1) = A_2(R'_1)$.*

Proof : Assume on the contrary that there exists $z \in A_2(R_1)$ such that $z \notin A_2(R'_1)$. We construct a preference R_2 as follows. On one hand, since from lemma 4, $A_2(R'_1)$ is closed, there exists a ball $B(z, \varepsilon)$ where $\varepsilon > 0$ such that $B(z, \varepsilon) \cap A_2(R'_1) = \varnothing$. On the other hand, from Lemma 1, we deduce that $x \in A_2(R'_1)$. Let :

$$w \equiv \text{Boundary } B(z, \varepsilon) \cap [x, z]$$

Since R_1 is strictly convex : wP_1z. Since R_1 is continuous, we deduce therefore that there exists a ball $B(w, \delta)$ where $\frac{\varepsilon}{2} > \delta > 0$ such that for all $u \in B(w, \delta)$: uP_1z. Let $\{u', u''\} \equiv$ Boundary $B(z, \varepsilon)\cap$ Boundary $B(w, \delta)$. Consider the two half- lines with origin x and going respectively through u' and u'' and the convex set S as on figure 7.

Proceeding as in the proof of Lemma 6, let H be defined over X as the gauge of S with respect to z and R_2 be the preference generated by H. We deduce that R_2 is strictly convex. Further, z is the unique best element and the boundary of S is the indifference curve going through x.

Since, by assumption, $z \in A_2(R_1)$, there exists $R'_2 \in D$ such that $z = C(R_1, R'_2)$. Since C is strategyproof, we deduce therefore that :

$$C(R_1, R_2) = z \tag{2}$$

Now, let B be the set of best alternatives of R_2 over $A_2(R_1')$. By construction of $B(z, \varepsilon)$, $B \cap B(z, \varepsilon) = \varnothing$. Also, by construction of S and since $x \in A_2(R_1') : B \subset S$. Further, since C is strategyproof, we deduce therefore that there exist $b \in B$ such that :

$$C(R_1', R_2) = b \tag{3}$$

From the construction of S and the position of b in S, we deduce from the strict convexity of R_1 that bP_1z. Indeed, b is necessarily such that $b = \lambda v + (1 - \lambda)\, x$ for some $\lambda \in [0, 1]$ and $v \in B(w, \delta)$. Therefore, since vP_1z and xP_1z, we deduce from the strict convexity of R_1 that bP_1z. Comparing (2) and (3), this implies then that C is manipulable by individual 1 at the profile $\pi = (R_1, R_2)$ in contradiction to our assumption that C is strategyproof $\square$

· We are now in position to prove the main result of this section. To proceed, we will use a result proved by Bordes, Laffond and Le Breton (1990) for the domain of Euclidean preferences over X. Let $\widehat{D}$ be the subset of Euclidean preferences over $\Re^2$ such that their ideal point belongs to X. Without any risk of confusion, we identify $\widehat{D}$ with X.

· **Theorem 2** *Let $m = 3$ and $n = 2$. Let C be a coalitional strategyproof social choice mechanism over $\widehat{D}$ such that $C(\widehat{D}^n) = X$. Then, C is dictatorial.*

· **Theorem 3** *Let $m = 3$ and $n = 2$. Let C be a strategyproof social choice mechanism over D such that $C(D^n) = X$. Then, C is dictatorial.*

Proof : From Lemma 7, D is rich and therefore, from theorem 1, C is coalitional strategyproof. Let $\widehat{C}$ be the restriction of C to $\widehat{D}^n$. Then, $\widehat{C}$ is also coalitional strategyproof. We deduce from Theorem 2 that $\widehat{C}$ is dictatorial. Without loss of generality, let individual 1 be the dictator for $\widehat{C}$. We now prove that 1 is also a dictator for C. This is equivalent to show that for all $\pi \in D^2$, the option set $A_2(R_1)$ is equal to the unique best element of R_1.

Let :

$$\widehat{A}_2(R_1) \equiv \left\{ x \in X : x = C(R_1, R_2) \text{ for some } R_2 \in \widehat{D} \right\}$$

From Lemma 3, we deduce that if $z = C(R_1, R_2)$, then $z = C(R_1, R_2')$ where $R_2' \in \widehat{D}_z$. From Lemma 6, $\widehat{A}_2(R_1) = A_2(x_1)$ where x_1 denotes both the best alternative for R_1 and the Euclidean preference with ideal point R_1. By combining both claims, we obtain that :

$$A_2(R_1) = A_2(x_1) = \widehat{A}_2(x_1)$$

But, since 1 is a dictator for C, $\widehat{A}_2(x_1) = \{x_1\}$ and the conclusion follows □

Single Peakedness

· Theorem 3 was derived under the assumption that there are at least three different public goods. When there are only two public goods, the set X is an interval. A preference in D over that interval is single peaked. We know that for this social environment there are many non dictatorial strategyproof social choice mechanisms, on top of which the so called median mechanism. The general family of strategyproof social choice mechanisms has been characterized by Moulin (1980). The domain D is rich and a shorter proof of the characterization result exploiting Theorem 1 could be provided. But more importantly, this setting is interesting as it illustrates the fact that there are domains where individual and coalitional strategyproofness are equivalent without being equivalent to dictatoriality.

· In fact, the richness property continues to hold in the case where single peakedness is defined with respect to an arbitrary tree instead of a segment, as in Demange (1982). The class of strategyproof social choice mechanisms operating over such a larger domain D of singlepeaked preferences has been characterized by Danilov (1994). Since D is rich, we deduce from Theorem 1 that these mechanisms are coalitional strategyproof. A direct proof of that assertion is provided by Danilov.

Piecewise Linear Preferences

· Theorem 3 can be extended to domains larger than D. Since here $X^* = X$, any social choice mechanism is trivially regular. From Lemma 4, we deduce that C is dictatorial iff C^* is dictatorial. Therefore, it is enough to prove that C^* is dictatorial. To do so, it is useful to observe again that in the proof of Lemma 6, we do not exploit the full force of the strict convexity of R_1. What is truly needed is the strict monotonicity along any half-line with the best alternative x_1 as origin, a property called star-shapedness by Border and Jordan (1983).

The structure of the proof of Theorem 3 is quite instructive. Once we know that the social choice mechanism C is coalitional strategyproof, we can exploit the simple fact that any restriction of C to a subdomain is also coalitional strategyproof. On these subdomains, the geometry of the Pareto set is sometimes easy to derive. For instance, in the case where the subdomain consists of the subset of Euclidean preferences, the Pareto set is the convex hull of the ideal points of the two individuals. The proof of Theorem 2 based on the technique of option sets uses this property. Once we know what happens on a subdomain, it remains of course to extend the result to the all domain. The key step, which corresponds here to Lemma 7, is a "top only" property (This "tops only" property is a familiar cornerstone in this area) asserting that strategyproofness implies that only the top alternatives of the two individuals matter in calculating the social outcome.

Theorem 3 is a slightly weaker version of an impossibility result established for this environment by Zhou (1991). His setting is identical to the one considered here but instead of us, Zhou does not assume that the range of the mechanism C coincides with X and demonstrates his result under the weaker assumption that the range of C is two dimensional.

To illustrate the usefulness of this method, we present a variant of the technique developed in the preceding section and derive a new result. Once again, after a proof that the domain we are going to consider is rich, it takes advantage of existing results on strategyproofness on a subdomain of that domain. Within the public good setting described in the preceding section, we consider now the set D of piecewise linear preferences over X. A preference R is piecewise linear if there exists a finite set of vectors $a^1, a^2, \ldots\ldots, a^K \in \Re^m$ such that for all $x, y \in X$:

$$xRy \text{ iff } Min\left(\left\langle a^1, x\right\rangle, \ldots\ldots, \left\langle a^K, x\right\rangle\right) \geq Min\left(\left\langle a^1, y\right\rangle, \ldots\ldots, \left\langle a^K, y\right\rangle\right)$$

· It can be verified that the domain D is rich. A proof similar to the proof used in the preceding section and based on the gauge function can be offered. An illustration of the argument behind this assertion is provided on figure 8.

· Let C be a regular and strategyproof social choice mechanism over D^n. Since D is rich, we deduce from Theorem 1 that it is coalitional strategyproof. This implies that its restriction $\widehat{C}$ to the subdomain of linear preferences is also coalitional strategyproof. From the results of Barbera and Gibbard, to which we already alluded, we deduce that $\widehat{C}$ is dictatorial. It remains to prove that C itself is dictatorial. To do so, it is enough to have a top only property similar to the one stated in Lemma 7 in the case of the preceding domain. We leave to the reader to check that a statement analogous to Lemma 7 holds true here: as already pointed out, a close look at the argument used in the proof shows that we do not exploit the full force of strict convexity.

Decomposability

The idea of decomposability applies when X is a product set $\prod_{k=1}^{m} X^k$ and preferences are individual preferences are separable : any joint preference R on X induces a profile of K

marginal $\left(R^k\right)_{1\leq k\leq m}$ preferences where R^k is a preference over X^k. A social choice mechanism is decomposable iff the for all $k=1,.....,m$ the k^{th} social choice depends exclusively upon the k^{th} profile of marginal $\left(R_i^k\right)_{1\leq i\leq n}$. With a decomposable mechanism, the decisions upon each coordinate are conducted separately. The preference information concerning the importance of the different coordinates for the players in the group is ignored by the mechanism

· If D consists exclusively of strict preferences and satisfies a mild regularity condition, then strategy-proofness implies decomposability (Le Breton and Sen (1999))

· When indifferences are permitted, things get much more complicated. Some partial decomposability results are avalaible (Border and Jordan (1983), Le Breton and Sen (1999), Le Breton and Weymark (1999))

The Spatial Model of Politics

· There are different versions of the spatial model differing according to their degree of generality (sse the references) concerning X the set of alternatives and D the domain of admissible preferences

· It is common to assume that X is a subset of some Euclidean space $\Re^m$

· Then, we may consider a nested sequence of domains D. Among many other possibilities, D can be:

- the set of continuous preferences with a (unique) bliss point

- the set of preferences R such that there exists a point x satisfying yRz iff $d(x,y)\leq d(x,z)$ where d is a fixed metric on X

- the set of continuous and separable preferences with a unique bliss point

- the set of continuous and strictly convex preferences

- the set of continuous, strictly convex and separable preferences

- the set of continuous and star-shaped preferences

- the set of continuous, star-shaped and separable preferences

- the set of quadratic preferences i.e; preferences R such that there exists a point x and a symmetric and positive definite matrix A such that yRz iff $(y-x)A(y-x)\leq(z-x)A(z-x)$

- the set of separable and quadratic preferences (A is diagonal)

- the set of Euclidean preferences (sometimes called type 1 preferences) (A is the identity)

- we could consider also arbitrary norms $\|.\|$ on $\Re^m$ and preferences R such that there exists a point x satisfying yRz iff $\|x-y\|\leq\|x-z\|$

Pareto-Optimality and Coalitional Strategy-Proofness in the Case of the Euclidean Domain

· Coalitional strategy-proofness (Bordes, Laffond and Le Breton (1990))

· Structure of the Pareto set and Pareto optimality (Kim and Roush (1984), Peters, Van der Stel and Storcken (1992, 1993a,b)) : $m = 2$ versus $m > 2$

· Pareto optimality and strategy-proofness when $m = 2$ and n is arbitrary; when $n = 2$ and m is arbitrary and when n and m are arbitrary

- Full Characterization for $m = 2$
- Impossibility Results for $m > 2$ when anonymity is added

Strategy-Proofness in the General Case

· We have already alluded to a weak form of Zhou's theorem.

· Border and Jordan decomposability 's theorem for separable quadratic and star shaped preferences

· Border and Jordan's impossibility results for the domain of quadratic preferences

Strategy-Proofness in the Euclidean Setting (Laffond (1980), Bordes Laffond and Le Breton (1990))

· Extensive study of the Euclidean setting when $n = 2$ and $m \geq 2$

· Characterization : An isomorphism between the class of unanimous, anonymous and strategy-proof mechanisms and the set of *self-dual cones* of $\Re^m$

· Classification of self-dual cones

· When $m \geq 3$, *there exist* non decomposable anonymous unanimous and strategy-proof social choice mechanisms

· Many open problems

Concluding Remarks

· What about the Bayesian version of this problem ?

· Here we have focused on private values. What about common values ?

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I have listed below a sample of useful references classified according to the main topics discussed during the seminar.

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